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Weak Cycle Partition Involving Degree Sum Conditions

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Abstract

Let G be a graph of order n and k a positive integer. A set of subgraphs $\mathcal{H}= \{H_1, H_2, \ldots, H_k\}$ is called a *k-degenerated cycle partition* (abbreviated k-DCP) of G if $H_1, \cdots, H_k$ are vertex disjoint subgraphs of G such that $V(G) = \bigcup_{i=1}^{k} V(H_i)$ and for all i, $1 \leq i \leq k$, H_i is a cycle or K_1 or K_2. If, in addition, for all i, $1 \leq i \leq k$, H_i is a cycle or K_1, then $\mathcal{H}$ is called a *k-weak cycle partition* (abbreviated k-WCP) of G. It has been shown by Enomoto and Li that if $|G| = n \geq k$ and if the degree sum of any pair of nonadjacent vertices is at least $n-k+1$, then G has a k-DCP. We prove that if G is a graph of order $n \geq k+12$ that has a k-DCP and if the degree sum of any pair of nonadjacent vertices is at least $\frac{3n+6k-5}{4}$, then either G has a k-WCP or $k = 2$ and G is a subgraph of $K_2 \cup K_{n-2} \cup \{e\}$, where e is an edge connecting $V(K_2)$ and $V(K_{n-2})$. By using this, we improve Enomoto and Li's result for $n \geq 10k+3$.

1 Introduction

In this paper, we only consider finite undirected graphs without loops and multiple edges. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. With a slight abuse of notation, for a subgraph H of G and a vertex $x \in V(G)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S \rangle$, and $G - S = \langle V(G) - S \rangle$. For a graph G, $|V(G)|$ is the order of G, $\delta(G)$ is the minimum degree of G, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) | x, y \in V(G), x \neq y, xy \notin E(G)\}$$

is the minimum degree sum of nonadjacent vertices. (When G is a complete graph, we define $\sigma_2(G) = \infty$.)

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If $C = c_1c_2\cdots c_pc_1$ is a cycle, we let $c_i\overrightarrow{C}c_j$, for $i \neq j$, be the subpath $c_ic_{i+1}\cdots c_j$, and $c_j\overleftarrow{C}c_i = c_jc_{j-1}\cdots c_i$, where the indices are taken modulo p. For any i and any $l \geq 2$, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, $c_i^{+l} = c_{i+l}$ and $c_i^{-l} = c_{i-l}$.

In this paper, "disjoint" means "vertex-disjoint," since we only deal with partitions of the vertex set.

Suppose $H_1, \cdots, H_k$ are disjoint subgraphs of G such that $V(G) = \bigcup_{i=1}^k V(H_i)$ and for all i, $1 \leq i \leq k$, H_i is a cycle or K_1 or K_2, then we call $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ a *k-degenerated cycle partition* (abbreviated k-DCP) of G. If, in addition, for all i, $1 \leq i \leq k$, H_i is a cycle, then the union of these H_i is a 2-factor of G with k components. A sufficient condition for the existence of a 2-factor with a specified number of components was given by Brandt et al. [1].

Theorem 1 *[1] Suppose $|G| = n \geq 4k$ and $\sigma_2(G) \geq n$. Then G can be partitioned into k cycles, that is, G contains k disjoint cycles $H_1, \cdots, H_k$ satisfying $V(G) = \bigcup_{i=1}^k V(H_i)$.*

In order to generalize 2-factors, Enomoto and Li [5] defined k-DCP by considering single edge and single vertex as degenerated cycles. They showed that weaker conditions than Theorem 1 are sufficient for the existence of k-DCP.

Theorem 2 *[5] Let G be a graph of order n and k any positive integer with $k \leq n$. If $\sigma_2(G) \geq n - k + 1$, then G has a k-DCP, except $G = C_5$ and $k = 2$.*

Note that a single vertex can be considered as a cycle of one vertex. Hu and Li [6] study the existence of a k-DCP $\{H_1, H_2, \ldots, H_k\}$, each of H_i is either a cycle or a single vertex. They defined such a k-DCP as a *k-weak cycle partition* (abbreviated k-WCP) of G. Firstly, they showed that under a weaker condition on degree sum, there is a k-DCP containing at most one K_2. Secondly, they showed that under a weaker condition on minimum degree, there is a k-WCP.

Theorem 3 *[6] Let G be a graph of order $n \geq k+12$ that has a k-DCP. If $\sigma_2(G) \geq \frac{2n+k-4}{3}$, then G has a k-DCP containing at most one subgraph isomorphic to K_2.*

Theorem 4 *[6] Let G be a graph of order n that has a k-DCP. If $\delta(G) \geq \frac{n+2k}{3}$, then G has a k-WCP.*

The graphs $G_t = mK_1 + (m+t)K_2$, $t \in \{1, 2\}$, show that both Theorem 3 and Theorem 4 are best possible. In this paper, we show that under a weaker condition on degree sum, there is a k-WCP.

Theorem 5 *Let G be a graph of order $n \geq k + 12$ that has a k-DCP. If $\sigma_2(G) \geq \frac{3n+6k-5}{4}$, then either G has a k-WCP or $k = 2$ and G is a subgraph of $K_2 \cup K_{n-2} \cup \{e\}$, where e is an edge connecting $V(K_2)$ and $V(K_{n-2})$.*

Note that $\sigma_2(K_2 \cup K_{n-2} \cup \{e\}) = n - 2$. By Theorem 2 and Theorem 5, we get

Theorem 6 *Suppose G is a graph of order $n \geq 10k + 3$. If $\sigma_2(G) \geq n - k + 1$, then G has a k-WCP.*

2 Proof of Theorem 5

Let G be a graph that satisfies the condition of Theorem 5. Since a 1-DCP is a hamiltonian cycle, Theorem 5 is true for $k=1$. Suppose $k \geq 2$. Then, $\sigma_2(G) \geq \frac{3n+6k-5}{4} \geq \frac{2n+k-4}{3}$. By Theorem 3, G has a k-DCP containing at most one subgraph isomorphic to K_2. Among all of these partitions, choose one, say $\mathcal{H}$, such that $c(\mathcal{H})$, the number of cycles in $\mathcal{H}$, achieves the minimum.

Let us suppose, to the contrary, that Theorem 3 is false. Then, $\mathcal{H}$ contains exactly one subgraph isomorphic to K_2. Denote $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ so that $H_1 = uv$ is a K_2 of G. Set

$$A = \{v \in V(G) : v \text{ is not in any cycle of } \mathcal{H}\},$$

and

$$B = \{v \in V(G) : v \text{ is in some cycle of } \mathcal{H}\}.$$

Then, $V(G) = A \cup B$ and

(2.1) $|A| = k - c(\mathcal{H}) + 1$.

Since $n \geq k+12$, by (2.1), $B \neq \emptyset$ and hence $\mathcal{H}$ contains at least one cycle. Let C be any cycle in $\mathcal{H}$. We first have

(2.2) $N_C^{++}(u) \cap N_C(v) = \emptyset$.

To justify (2.2), we assume, to the contrary, that $x \in N_C^{++}(u) \cap N_C(v)$. Set $C^{(1)} = x\overrightarrow{C}x^{--}uvx$. Then, $(\mathcal{H} \setminus \{C, H_1\}) \cup \{C^{(1)}, x^-\}$ is a k-WCP of G. Hence, (2.2) is true.

Similarly, we have

(2.3) For every $w \in A$, $N_C(w) \cap N_C^+(w) = \emptyset$.

We consider the following two cases.

Case 1. $\min\{d_B(u), d_B(v)\} > 0$.

Case 1.1. There exists a cycle C in $\mathcal{H}$ such that either $N_C^+(u) \cap N_C(v)$ or $N_C^-(u) \cap N_C(v)$ is not empty.

By symmetry, we may assume that $N_C^+(u) \cap N_C(v) \neq \emptyset$. Let $x \in N_C^+(u) \cap N_C(v)$. If $x^{--} = x^+$, then $(\mathcal{H} \setminus \{C, H_1\}) \cup \{uvxx^-u, x^+\}$ is a k-WCP of G. Hence, $x^{--} \neq x^+$.

(2.4) $N_C^{++}(x) \cap N_C(x^+) \subseteq \{x\}$.

Suppose, to the contrary, that $y \in (N_C^{++}(x) \cap N_C(x^+)) \setminus \{x\}$. Then, $y \neq x^+, x^{++}$. Set $C^{(2)} = y\overrightarrow{C}x^-uvxy^{--}\overleftarrow{C}x^+y$. Then, $(\mathcal{H} \setminus \{C, H_1\}) \cup \{C^{(2)}, y^-\}$ is a k-WCP of G. Hence, (2.4) is true.

(2.5) $N_C(x^+) \cap N_C^+(v) = \emptyset$.

Indeed, if $y \in N_C(x^+) \cap N_C^+(v)$, then $y \neq x^+$. Set $C^{(3)} = y\overrightarrow{C}xvy^-\overleftarrow{C}x^+y$. Then, $(\mathcal{H} \setminus \{C, H_1\}) \cup \{C^{(3)}, u\}$ is a k-WCP of G. Hence, (2.5) is true.

(2.6) $N_C^{++}(x) \cap N_C^+(v) \subseteq \{x^+, x^{+3}\}$.

Assume, to the contrary, that $y \in N_C^{++}(x) \cap N_C^+(v) \setminus \{x^+, x^{+3}\}$. Then, $y^- \in N_C(v)$. Since $x \in N_C(v)$, by (2.3), we have $y^- \neq x^+$. Set $C^{(4)} = y^-\overrightarrow{C}x^-uvy^-$ and $C^{(5)} = xy^{--}\overleftarrow{C}x$. Since $y^- \neq x, x^+, x^{++}$, $(\mathcal{H} \setminus \{C, H_1\}) \cup \{C^{(4)}, C^{(5)}\}$ is a k-WCP of G. Hence, (2.6) is true.

It follows from (2.4)–(2.6) that $d_C(x) + d_C(x^+) + d_C(v) \leq |C| + 3$. By symmetry, we also have $d_C(x^-) + d_C(x^{--}) + d_C(u) \leq |C| + 3$. Hence

(2.7) $d_C(x^{--}) + d_C(x^-) + d_C(x) + d_C(x^+) + d_C(u) + d_C(v) \leq 2|C| + 6$.

In the following, we let C' be any cycle in $\mathcal{H} \setminus \{C\}$ (if any).

(2.8) $N_{C'}(x^+) \cap N_{C'}^{+3}(v) = \emptyset$.

Suppose, to the contrary, that $y \in N_{C'}(x^+) \cap N_{C'}^{+3}(v)$. Set $C^{(6)} = x^+\overrightarrow{C}xvy^{-3}\overleftarrow{C'}yx^+$. Then, $(\mathcal{H} \setminus \{C, C', H_1\}) \cup \{C^{(6)}, u, y^-y^{--}\}$ is a k-DCP with one K_2 and with fewer cycles than $\mathcal{H}$, a contradiction. Hence, (2.8) is true.

(2.9) $N_{C'}(x^+) \cap N_{C'}^+(x^-) = \emptyset$.

To justify (2.9), assume, to the contrary, that $y \in N_{C'}(x^+) \cap N_{C'}^+(x^-)$. Set $C^{(7)} = x^+\overrightarrow{C}x^-y^-\overleftarrow{C'}yx^+$. Then, $(\mathcal{H} \setminus \{C, C'\}) \cup \{C^{(8)}, x\}$ is a k-DCP with one K_2 and with fewer cycles than $\mathcal{H}$, a contradiction. Hence, (2.9) is true.

(2.10) $N_{C'}^+(x^-) \cap N_{C'}^{+3}(v) = \emptyset$.

Suppose, to the contrary, that $y \in N_{C'}^+(x^-) \cap N_{C'}^{+3}(v)$. Set $C^{(8)} = x\overrightarrow{C}x^-y^-\overrightarrow{C'}y^{-3}vx$. Then, $(\mathcal{H} \setminus \{C, C', H_1\}) \cup \{C^{(8)}, u, y^{--}\}$ is a k-WCP of G. Hence, (2.10) is true.

It follows from (2.8)–(2.10) that

$$d_{C'}(x^-) + d_{C'}(v) + d_{C'}(x^+) \leq |C'|$$

By symmetry, we also have

$$d_{C'}(x) + d_{C'}(u) + d_{C'}(x^{--}) \leq |C'|.$$

Hence,

(2.11) $d_{C'}(x^{--}) + d_{C'}(x^-) + d_{C'}(x) + d_{C'}(x^+) + d_{C'}(u) + d_{C'}(v) \leq 2|C'|.$

By (2.7) and (2.11), we get

(2.12) $\quad d_B(x^{--}) + d_B(x^-) + d_B(x) + d_B(x^+) + d_B(u) + d_B(v) \leq 2|B| + 6.$

Recall that $|A| = k - c(\mathcal{H}) + 1$. To avoid a k-WCP, we have $N_A(x^-) \cap N_A(x^{--}) = N_A(x) \cap N_A(x^+) = \emptyset$. This together with $u, v \in A$ implies

$$d_A(x^{--}) + d_A(x^-) + d_A(x) + d_A(x^+) + d_A(u) + d_A(v) \leq 2|A| + 2(|A| - 1).$$

Combining this with (2.12), we get

$$\begin{aligned} & d_G(x^{--}) + d_G(x^-) + d_G(x) + d_G(x^+) + d_G(u) + d_G(v) \\ \leq \;& (4|A| - 2) + (2|B| + 6) \\ = \;& 2n + 2|A| + 4. \end{aligned}$$

This together with (2.1) and $\sigma_2(G) \geq \frac{3n+6k-5}{4}$ implies

(2.13) $d_G(x^{--}) + d_G(x^-) + d_G(x) + d_G(x^+) + d_G(u) + d_G(v) < 3\sigma_2(G).$

Recall that $x \in N_C^+(u) \cap N_C(v)$. By (2.3), we have $xu, x^-v \notin E(G)$. Hence, $d_G(x^-) + d_G(x) + d_G(u) + d_G(v) \geq 2\sigma_2(G)$. Combining this with (2.13), we get $d_G(x^{--}) + d_G(x^+) < \sigma_2(G)$. Hence, $x^{--}x^+ \in E(G)$.

(2.14) $|C| = 4.$

Suppose, to the contrary, that $|C| \geq 5$. Set $C^{(9)} = x^+\overrightarrow{C}x^{--}x^+$ and $C^{(10)} = uvxx^-u$. Then, $(\mathcal{H}\backslash\{C, H_1\}) \cup \{C^{(9)}, C^{(10)}\}$ is a k-WCP of G. This contradiction proves (2.14).

It follows from (2.14) that $|V(C) \cup V(H_1)| = 6$. To avoid a k-WCP, $\langle V(C) \cup V(H_1)\rangle$ contains no cycle of length 5. Hence, $x^{--}u$, x^+v, $x^{--}x$, x^+x^-, x^-v, $xu \notin E(G)$. This implies

$$2[d_G(x^{--}) + d_G(x^-) + d_G(x) + d_G(x^+) + d_G(u) + d_G(v)] \geq 6\sigma_2(G),$$

contrary to (2.13). This contradiction completes the proof of Case 1.1.

Case 1.2. For every cycle C in $\mathcal{H}$, $N_C^+(u) \cap N_C(v) = N_C^-(u) \cap N_C(v) = \emptyset$.

Let C be any cycle in $\mathcal{H}$. By (2.2), (2.3) and the assumption of this case, we see that $N_C^{++}(u), N_C^{+}(u), N_C(v)$ are pairwise disjoint sets of $V(C)$. Hence, $2d_C(u)+d_C(v)\leq |C|$. By symmetry, we also have $2d_C(v)+d_C(u)\leq |C|$. Therefore, $d_C(u)+d_C(v)\leq \frac{2|C|}{3}$. This together with the definition of B implies

$$d_B(u)+d_B(v)\leq \frac{2|B|}{3}.$$

On the other hand, by $u,v\in A$ and (2.1), we have

$$d_A(u)+d_A(v)\leq 2(|A|-1)\leq \frac{2|A|}{3}+\frac{4k}{3}-2,$$

and hence

$$d_G(u)+d_G(v)\leq (\frac{2|A|}{3}+\frac{4k}{3}-2)+\frac{2|B|}{3}=\frac{2n+4k-6}{3}.$$

By $\sigma_2(G)\geq \frac{3n+6k-5}{4}$, we get

(2.15) $d_G(u)+d_G(v)<\sigma_2(G)$.

(2.16) $N_C^{+3}(u)\cap N_C(v)=\emptyset$.

Assume, to the contrary, that $w\in N_C^{+3}(u)\cap N_C(v)$. Then, by (2.3), we have $uw^{--}, vw^{-}\notin E(G)$, and hence

$$d_G(u)+d_G(w^{--})+d_G(v)+d_G(w^{-})\geq 2\sigma_2(G).$$

Set $C^{(11)}=w\overrightarrow{C}w^{-3}uvw$ and $\mathcal{H}'=(\mathcal{H}\setminus\{C,H_1\})\cup\{C^{(11)},w^{-}w^{--}\}$. Then, $\mathcal{H}'$ is a k-DCP of G containing only one K_2 and $c(\mathcal{H}')=c(\mathcal{H})$. So, $\mathcal{H}'$ and $w^{-}w^{--}$ play a similar role as $\mathcal{H}$ and uv. Note that $w^{-3}\in N_{C^{(11)}}(w^{--})$ and $w\in N_{C^{(11)}}(w^{-})$. By an argument similar to that in the proof of Case 1.1 and (2.15), we can derive that $d_G(w^{--})+d_G(w^{-})<\sigma_2(G)$. This together with (2.15) implies

$$d_G(u)+d_G(w^{--})+d_G(v)+d_G(w^{-})<2\sigma_2(G),$$

a contradiction. Therefore, (2.16) is true.

It follows from (2.2), (2.3), (2.16) and the assumption of Case 1.2 that $N_C^{++}(u)$, $N_C^{+3}(u)$, $N_C(v)$ and $N_C^{+}(v)$ are pairwise disjoint subsets of $V(C)$. Hence, $2d_C(u)+2d_C(v)\leq |C|$ implying that $2d_B(u)+2d_B(v)\leq |B|$. Since $V(G)=A\cup B$, by (2.1), we get

(2.17) $d_G(u)+d_G(v)\leq 2(|A|-1)+\frac{|B|}{2}=\frac{n+3k-3c(\mathcal{H})-1}{2}$.

It follows from the assumption of Case 1 that there exists a cycle C in $\mathcal{H}$ so that $N_C(u)\neq\emptyset$. Similarly, there exists a cycle C' in $\mathcal{H}$ so that $N_{C'}(v)\neq\emptyset$. Let $x\in N_C(u)$ and $y\in N_{C'}(v)$. By (2.3), we have

(2.18) $x^-u, y^+v \notin E(G)$.

We consider the following two subcases.

Case 1.2.1. $C = C'$.

It follows from the assumpation of Case 1.2 that $x \neq y^+$. Note that $x\overrightarrow{C}yvux$ is a cycle in $\langle V(C) \cup V(H_1)\rangle$. To avoid a k-WCP, we have $x^- \neq y^+$ and $\langle y^+\overrightarrow{C}x^-\rangle$ contains no hamiltonian cycle. By standard arguments on hamiltonian graph theory, we can derive that

(2.19) $d_{y^+\overrightarrow{C}x^-}(x^-) + d_{y^+\overrightarrow{C}x^-}(y^+) \leq |y^+\overrightarrow{C}x^-|$.

(2.20) For every cycle C'' in $\mathcal{H} \setminus \{C\}$, $d_{C''}(x^-) + d_{C''}(y^+) \leq |C''|$.

Indeed, if (2.20) is false, then there is a vertex $z \in V(C'')$ so that $x^-z^-, y^+z^+ \in E(G)$. Set $C^{(13)} = x\overrightarrow{C}yvux$ and $C^{(14)} = y^+\overrightarrow{C}x^-z^-\overleftarrow{C''}z^+y^+$. Then, $(\mathcal{H}\setminus\{H_1, C, C''\}) \cup \{C^{(13)}, C^{(14)}, z\}$ is a k-WCP of G. Hence, (2.20) is true.

By replacing C'' with $C^{(13)}$ in the proof of (2.20), we get

(2.21) $d_{C^{(13)}}(x^-) + d_{C^{(13)}}(y^+) \leq |C^{(13)}|$.

It follows from (2.19) and (2.21) that $d_{C\cup H_1}(x^-) + d_{C\cup H_1}(y^+) \leq |C| + 2$. This together with (2.20) and the definition of B implies that $d_{B\cup H_1}(x^-) + d_{B\cup H_1}(y^+) \leq |B| + 2$. Since $d_{A\setminus V(H_1)}(x^-) + d_{A\setminus V(H_1)}(y^+) \leq 2(|A| - 2) = |A| + k - c(\mathcal{H}) - 3$, we have $d_G(x^-) + d_G(y^+) \leq (|A| + k - c(\mathcal{H}) - 3) + (|B| + 2) = n + k - c(\mathcal{H}) - 1$. This together with (2.17) implies $d_G(u) + d_G(v) + d_G(x^-) + d_G(y^+) \leq \frac{3n+5k-5c(\mathcal{H})-3}{2} < 2\sigma_2(G)$. Hence, $\{x^-u, y^+v\} \cap E(G) \neq \emptyset$, contrary to (2.18). This contradiction completes the proof of Case 1.2.1.

Case 1.2.2. $C \neq C'$.

In this case, we have $c(\mathcal{H}) \geq 2$. Set $P = x^-\overleftarrow{C}xuvy\overleftarrow{C'}y^+$. Then, P is a hamiltonian path of $\langle V(C) \cup V(C') \cup V(H_1)\rangle$. To avoid a k-WCP, $\langle V(C) \cup V(C') \cup V(H_1)\rangle$ contains no cycle of length $|V(P)| - 2$, and so $N_P(x^-) \cap N_P^{+3}(y^+) = \emptyset$. This implies

(2.22) $d_P(x^-) + (d_P(y^+) - 2) \leq |V(P)|$.

(2.23) If C'' is a cycle of $\mathcal{H}\setminus\{C, C'\}$ with length at least 4, then $d_{C''}(x^-) + d_{C''}(y^+) \leq |C''|$.

Indeed, if (2.23) is false, then there is a vertex $z \in V(C'')$ so that $x^-z^{--}, y^+z^{++} \in$

$E(G)$. Set $C^{(15)} = x^-\overrightarrow{P}y^+z^{++}\overrightarrow{C''}z^{--}x^-$. Then, $(\mathcal{H}\backslash\{H_1, C, C', C''\})\cup\{C^{(15)}, z^-, z, z^+\}$ is a k-WCP of G. Hence, (2.23) is true.

Note that for every cycle C'' of length 3, $d_{C''}(x^-) + d_{C''}(y^+) \leq |C''| + 3$. By (2.22) and (2.23), we have

$$d_{B\cup V(H_1)}(x^-) + d_{B\cup V(H_1)}(y^+) \leq (|B \cup V(H_1)| + 2) + 3(c(\mathcal{H}) - 2).$$

On the other hand, by $|A \setminus V(H_1)| = k - c(\mathcal{H}) - 1$, we have

$$d_{A\setminus V(H_1)}(x^-) + d_{A\setminus V(H_1)}(y^+) \leq |A \setminus V(H_1)| + (k - c(\mathcal{H}) - 1).$$

Hence, $d_G(x^-) + d_G(y^+) \leq n + k + 2c(\mathcal{H}) - 5$. This together with (2.17) implies

$$d_G(u) + d_G(v) + d_G(x^-) + d_G(y^+) \leq \frac{3n + 5k + c(\mathcal{H}) - 11}{2} < 2\sigma_2(G).$$

Hence, $\{x^-u, y^+v\} \cap E(G) \neq \emptyset$, contrary to (2.18). This contradiction completes the proof of Case 1.2.2. The proof of Case 1 is completed.

Case 2. $\min\{d_B(u), d_B(v)\} = 0$.

We may assume, without loss of generality, that $d_B(v) = 0$. Then, $d_G(v) = d_A(v) \leq |A| - 1 \leq k - 1$. By the degree sum condition, we have

(2.24) For every $x \in B$, $d_G(x) \geq \sigma_2(G) - d_G(v) \geq \frac{3n+2k-1}{4}$.

(2.25) $c(\mathcal{H}) = 1$.

Suppose, to the contrary, that (2.25) is false, then $c(\mathcal{H}) \geq 2$. Let C be a cycle in $\mathcal{H}$ with minimum length and let $x \in V(C)$. Note that $|A| = k - c(\mathcal{H}) + 1$. To avoid a k-WCP, we have for every cycle C' in $\mathcal{H} \setminus \{C\}$ that $\langle V(C) \cup V(C')\rangle$ contains no hamiltonian cycle. This implies $N^+_{C'}(x^-) \cap N_{C'}(x) = \emptyset$, and hence $d_{C'}(x^-) + d_{C'}(x) \leq |C'|$. Since $d_C(x^-) + d_C(x) \leq 2(|C| - 1) = |C| + (|C| - 2)$, by the definition of B, we have

$$d_B(x^-) + d_B(x) \leq |B| + |C| - 2 \leq \frac{3|B| - 4}{2}.$$

This together with $d_A(x^-) + d_A(x) \leq 2|A|$ implies

$$\begin{aligned} d_G(x^-) + d_G(x) &\leq 2|A| + \frac{3|B| - 4}{2} \\ &= \frac{3n + |A| - 4}{2} \\ &\leq \frac{3n + k - c(\mathcal{H}) - 3}{2}, \end{aligned}$$

contrary to (2.24). Hence, (2.25) is true.

It follows from (2.1) and (2.25) that $|A| = k$. In the following, we let C be the only cycle in $\mathcal{H}$. Clearly, $V(C) = B$. Since $u, v \in A$, we have the following two subcases.

Case 2.1. $|A| \geq 3$.

Let $w \in A \setminus \{u, v\}$, then there exists an integer i, $2 \leq i \leq k$, so that $V(H) = \{w\}$. By (2.3), $N_C(w) \cap N_C^+(w) = \emptyset$ and hence

$$d_G(w) = d_A(w) + d_C(w) \leq (|A| - 1) + \frac{|C|}{2} = \frac{n + k - 2}{2}.$$

This together with $d_G(v) = d_A(v) \leq k - 1$ implies

$$d_G(v) + d_G(w) \leq \frac{n + 3k - 4}{2} < \sigma_2(G).$$

Hence

(2.26) $vw \in E(G)$.

(2.27) $uw \notin E(G)$.

To justify (2.27), we assume to the contrary that $uw \in E(G)$. Then, $C^{(16)} = uvwu$ is a cycle of G. Note that $|A| = k - c(\mathcal{H}) + 1 = k$. By $n \geq k + 12$, we have $|C| = |B| \geq 12$. Let x be any vertex in C. By (2.3), we have $N_A(x) \cap N_A(x^-) = \emptyset$. Hence, $d_A(x) + d_A(x^-) \leq |A| = k$. This together with (2.24) implies

$$\begin{aligned} d_C(x) + d_C(x^-) &= (d_G(x) + d_G(x^-)) - (d_A(x) + d_A(x^-)) \\ &\geq \frac{3n + 2k - 1}{2} - k. \end{aligned}$$

Hence, $N_C^{++}(x^-) \cap N_C(x) \neq \emptyset$. Let $y \in N_C^{++}(x^-) \cap N_C(x)$. Define $C^{(17)} = x\overrightarrow{C}y^{--}x^-\overleftarrow{C}yx$. Then, $(\mathcal{H} \setminus \{H_1, H_i, C\}) \cup \{C^{(16)}, C^{(17)}, y^-\}$ is a k-WCP of G. This contradiction completes the proof of (2.27).

(2.28) $N_C^{+3}(u) \cap N_C(w) = \emptyset$.

Suppose, to the contrary, that $y \in N_C^{+3}(u) \cap N_C(w)$. Set $C^{(18)} = y\overrightarrow{C}y^{-3}uvwy$. Then, $(\mathcal{H} \setminus \{H_1, H_i, C\}) \cup \{C^{(18)}, y^-, y^{--}\}$ is a k-WCP of G. Hence, (2.28) is true.

(2.29) $N_C^{+4}(u) \cap N_C(w) = \emptyset$.

To justify (2.29), we assume by contradiction that $y \in N_C^{+4}(u) \cap N_C(w)$. Set $C^{(19)} = y\overrightarrow{C}y^{-4}uvwy$ and $\mathcal{H}' = (\mathcal{H} \setminus \{H_1, H_i, C\}) \cup \{C^{(19)}, y^-y^{--}, y^{-3}\}$. Then, $\mathcal{H}'$ is a k-DCP with $c(\mathcal{H}') = 1$ and with one subgraph isomorphic to K_2. Clearly, $N_{C^{(19)}}(y^-) \neq \emptyset$. On the other hand, by (2.24), we have $d_C(y^{--}) = d_G(y^{--}) - d_A(y^{--}) \geq \frac{3n+2k-1}{4} - k > \frac{n+23}{4}$. This together with $|V(C) \setminus V(C^{(19)})| = 3$ implies $N_{C^{(19)}}(y^{--}) \neq \emptyset$. Hence, the pair $(\mathcal{H}', y^-y^{--})$ plays a similar role as $(\mathcal{H}, uv)$ in Case 1. By an argument similar to that in the proof of Case 1, we can get a contradiction. Hence, (2.29) is true.

It follows from (2.3), (2.28) and (2.29) that $N_C^{+3}(u)$, $N_C^{+4}(u)$ and $N_C(w)$ are pairwise disjoint subsets of $V(C)$. Hence, $2d_C(u) + d_C(w) \leq |C|$. By symmetry, we also have $2d_C(w) + d_C(u) \leq |C|$. Hence, $d_C(u) + d_C(w) \leq \frac{2|C|}{3}$ This together with $|C| = n - |A| = n - k$ implies

$$\begin{aligned} d_G(u) + d_G(w) &= (d_A(u) + d_A(w)) + (d_C(u) + d_C(w)) \\ &\leq 2(|A| - 2) + \frac{2|C|}{3} \\ &= \frac{2n + 4k - 12}{3}. \end{aligned}$$

Hence, $d_G(u) + d_G(w) < \sigma_2(G)$, contrary to (2.27). This contradiction completes the proof of Case 2.1.

Case 2.2. $|A| = 2$.

In this case, we have $k = 2$, $A = \{u, v\}$ and $d_C(v) = 0$. To prove the Theorem, it suffices to show that $d_C(u) \leq 1$. Assume, to the contrary, that $d_C(u) \geq 2$. Let x and y be two distinct neighbors of u in C. By (2.24) and (2.25), we have for every $z \in V(C)$

$$d_C(z) = d_G(z) - d_A(z) \geq \frac{3n + 2k - 1}{4} - 1 \geq \frac{2n + 3k + 7}{4} > \frac{|C|}{2}.$$

Hence, $\langle V(C) \rangle$ is hamiltonian connected. In particular, there is a hamiltonian (x, y)-path P in $\langle V(C) \rangle$. Let $C^{(19)} = x\overrightarrow{P}yux$. Then, $\{C^{(19)}, v\}$ is a 2-WCP of G. This contradiction completes the proof of Case 2.2 and hence Theorem 5 is proved. □

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