A note on the number of triangles in C_{2k+1}-free graphs

Ervin Győri *

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest H-1364, POB 127, Hungary
e-mail: ervin@renyi.hu

Hao Li †

Laboratoire de Recherche en Informatique
UMR 8623, C.N.R.S.-Université de Paris-sud
91405-Orsay cedex, FRANCE
e-mail: li@lri.fr

Abstract

Upper and lower bounds are proved for the maximum number of triangles in C_{2k+1}-free graphs. The bounds involve extremal numbers related to appropriate even cycles.

1 Introduction and notation

Throughout the paper, we follow the usual notation you can find, say, in [2].

Erdős [5] stated several conjectures in extremal graph theory related to triangles and pentagons. We recall just the most relevant one.

Conjecture 1. The number of cycles of length in a triangle-free graph of order n is at most $(n/5)^5$ and equality holds for the blown-up pentagon if $5|n$.

The best published upper bound about $1.03(n/5)^5$ is proved in [7], but Füredi announced an improvement to $1.01(n/5)^5$ or maybe to $1.001(n/5)^5$.

Bollobás and Győri [[1]] studied the natural, less studied converse of the problem: what can we say about the number of triangles in a graph not containing any pentagon.

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†Corresponding author. The work was partially supported by NNSF of China (60373012)

Theorem 1. If G is a graph not containing any C_5 then the number of triangles in G is at most $(\sqrt{2}/4+1)n^{3/2}+o(n^{3/2})$.

Theorem 1 is sharp apart from the constant coefficient as the following example shows.

Example 1. Let G_0 be a C_4-free bipartite graph on $n/3+n/3$ vertices with about $(n/3)^{3/2}$ edges. Double each vertex in one of the color classes and add an edge joining the old and the new copy. (We call these edges monochromatic.) Let G denote the resulting graph. The number of edges in G is $2(n/3)^{3/2}+o(n^{3/2})$. Clearly, the number of triangles in G is the number of edges in G_0 and G does not contain any C_5.

In this paper, we generalize Theorem 1 and Example 1 replacing the pentagons with longer odd cycles. Interestingly, the number of triangles in a C_{2k+1}-free graph is bounded by different constant times the extremal edge numbers in graphs not containing C_{2k} or having girth $2k+2$. Let us remark that it again calls our attention to the old and classical question: how close are the functions $ex(n;C_{2k}$ and $ex(n;C_4,C_6,...,C_{2k}$ to each other.

The main theorem we are to prove is as follows.

<u>Theorem 2</u>: For any integer $k\geq 2$, if G is a C_{2k+1}-free simple graph. Then the number $t(G)$ of the triangles in G is less than $\frac{(2k-1)(16k-2)}{3}ex(n,C_{2k})$.

Remark. The upper bound in Theorem 2 is essentially sharp. The following example shows that there exists a graph G such that

$$t(G)\geq\binom{k}{2}ex_{bip}(\frac{2n}{k+1};C_4,C_6,...,C_{2k}).$$

If we assume that the function $ex_{bip}(n;C_4,C_6,...,C_{2k})$ behaves nicely, like say, n^c then it implies easily that $t(G)\geq(k-2)ex_{bip}(n;C_4,C_6,...,C_{2k})$. (The estimate $t(G)\geq ex_{bip}(n;C_4,C_6,...,C_{2k})$ can be proved easily without any assumption.) Since the functions $ex_{bip}(n;C_4,C_6,...,C_{2k}), ex(n;C_4,C_6,...,C_{2k}), ex(n;C_{2k})$ are essentially the same ([**?**]), it follows that our estimate in Theorem 2 is essentially sharp.

Example. Take a maximum size bipartite graph $H(X_0,Y)$ with $|X_0|=\frac{n}{k+1},|Y|=\frac{n}{k+1}$ such that $C_4,C_6,...,C_{2k}\nsubseteq H$. To get the desired graph G, "blow up" the vertices in X, more precisely for every vertex $x\in X$, replace x by k vertices $x_1,x_2,...x_k$ joined to each other and all neighbors of x. The set of these new vertices is denoted by X and the resulting graph G has n vertices. This graph G contains many cycles of length $3,4,...,2k$: Take k neighbors of a vertex $x\in X_0$ in H, then $x_1,x_2,...,x_k$ and these neighbors constitute a split graph, i.e. a $K_{k,k}$ plus all the edges in one of the color classes. But suppose that G contains a $(2k+1)$-cycle C. Since Y is independent, C contains at least $k+1$ vertices in X. Now, contract the cliques of the vertices $x_1,...,x_k$ for every vertex $x\in X_0$ to get back the graph H and lat what happens to C. The cycle C is transformed into a closed walk C' in the bipartite graph H which contains at least two vertices in X_0 and uses every vertex in Y only once. So, there is a vertex $y\in V(C')\cap Y$ such that the neighbors of y in C' are distinct and C' contains an even cycle of length at most $2k+1$, which is contradiction.

Now count the triangles in G. The set Y is independent, so every triangle contains at least two vertices in X. The number of triangles containing two and three vertices in X is $\binom{k}{2}e(G)$ and $\binom{k}{3}\frac{n}{k+1}$, respectively. The second term is linear in n, so it is neglectable.

2 Preliminary Lemmas

We will prove the main theorem by showing the following two results:

Lemma A: If G is a C_{2k+1}-free simple graph such that every edge is in at least one triangle. Then the number $t(G)$ of the triangles in G is at most $\frac{(2k-2)e(G)}{3}$.

Proof of Lemma A: For any vertex x, the number t_x of triangles containing x is $e(G[N(x)])$, the number of edges in $N(x)$. Since G is a C_{2k+1}-free, $G[N(x)]$ does not contain any path of $2k$ vertices. So,

$$t_x \leq (k-1)d(x)$$

by the classical theorem of Erdős and Gallai [eg]. By adding up these inequalities, it follows that $t(G) = \frac{1}{3}\sum_{x\in V(G)} t_x \leq \frac{2k-2}{3}e(G)$, as required.

□

Lemma B: If G is a C_{2k+1}-free simple graph such that every edge is in at least one triangle. Then $e(G)$ is at most $(16k-2)ex(n, C_{2k})$.

Proof of Lemma B: Put $G_0 = G$ and let us define three sets by

$R_0 := \emptyset$,

$W_0 := e(G)$ and

$D_0 := \emptyset$.

Suppose that we have defined G_i together with R_i, W_i, D_i, $i \geq 0$.

We call an edge in R_i, W_h and D_h *red, white* and *deleted* edge in G_h, respectively. For a vertex x we denote by p_x the number of white edges incident to x in G_i and by q_x the number of white edges in the neighborhood subgraph $G[(N(x))]$.

Choose a vertex x such that $8k\, q_x < p_x$ if there is such one and we define the followings:

$G_{i+1} = G_i - \{e \in W_i\colon\ e \text{ is incident to } x\}$,

$D_{i+1} = D_i \cup \{e \in W_i\colon\ e \text{ is incident to } x\}$,

$W_{i+1} = W_i - \{e \in W_i\colon\ e \text{ is incident to } x\} - \{e \in W_i \cap G[(N(x)])\}$ and

$R_{i+1} = R_i \cup \{e \in W_h \cap G[N(x)]\}$.

Claim 1: There is no $2k$-cycle C in G_i such that $E(C) \cap R_i \neq \emptyset$ for $i = 0, 1, 2, ...$

Proof of Claim 1: We prove the claim by induction on i.

If $i = 0$ then we have nothing to prove.

Assume that there is no $2k$-cycle in $G_0, G_1, ..., G_i$ that contains any edge in $R_0, R_1, ..., R_i$, respectively. Suppose that there is a $2k$-cycle C in G_{i+1} that contains at least one edge in R_{i+1}. Then by the inductional hypothesis, this cycle C contains an edge f in $R_{i+1} - R_i = \{e \in W_i \cap G[N(x)]\}$. Every edge incident to x in G_{i+1} is in R_i and by the inductional hypothesis again, it is not in $E(C)$. Therefore x is not in $V(C)$. Since the edge f is in $N(x)$. Let e_1, e_2 be the two edges incident to the end vertices of f and x respectively. Then $(C - \{f\}) \cup \{e_1, e_2\}$ is a $(2k+1)$-cycle in G, a contradiction. So there is no $2k$-cycle in G_{i+1} that contains at least one edge in R_{i+1}.

□

Let j be the smallest index such that , $8kq_x \leq p_x$ in G_j for every vertex x, i.e. our procedure stops.

We distinguish two cases regarding W_j.

Case 1. $W_j = \emptyset$.

By Claim 1, we have $e(G_j) \leq ex(n, C_{2k})$.

According to the definition, we had $8kq_x > p_x$ in each step. It follows $|D_{i+1} - D_i| = p_x > 8kq_x$ for $i = 0, 1, ..., j-1$ and hence

$$\sum_{i=0}^{j} |D_i| \leq 8k|R_f| \leq \frac{1}{f(k)} ex(n, C_{2k}).$$

We obtain $e(G) \leq (1 + \frac{1}{f(k)}) ex(n, C_{2k})$.

Case 2: $W_j \neq \emptyset$**.**

We have $8kq_x \leq p_x \leq d(x)$ in G_j for every vertex x.

Since every edge of G is in at least one triangle, it follows that for any x,

$$|\{e \in W_j \cap G[(N_(x)]\}| + |\{e \in (R_j \cup D_j) \cap G[N(x)]\}| \geq \frac{d(x)}{2}$$

and hence

$$\begin{aligned} |\{e \in (R_j \cup D_j) \cap G[N(x)]\}| &\geq \tfrac{d(x)}{2} - \tfrac{1}{8k} d(x) \\ &\geq (4k-1)q_x. \end{aligned}$$

By Lemma A, the number of triangles in G is at most $\frac{2k-1}{3} e(G)$. But, every edge $e \in (R_j \cup D_j) \cap G[N(x)]\}$ is in a triangle containing x. This triangle could be counted at most 3 times when we count $\{e \in (R_j \cup D_j) \cap G[N(x)]\}$ for every x. It follows that

$$\begin{aligned} (2k-1)e(g) \geq 3t(G) &\geq \textstyle\sum_{x \in G} |\{e \in (R_j \cup D_j) \cap G[N(x)]\}| \\ &\geq (4k-1)q_x \\ &\geq (4k-1)|W_j|. \end{aligned}$$

Thus we get

$$|W_j| \leq \frac{2k-1}{4k-1)}e(G).$$

On the other hand, in each step i, we have $8kq_x \geq p_x$ and we know that $|D_{i+1} - D_i| = p_x < 8kq_x$ and hence $|D_j| \leq 8k\sum_{i=0}^{j} q_x = 8k|R_f|$. It follows that

$$\begin{aligned} \tfrac{2k}{4k-1}e(G) &\leq e(G) - |W_j| = |D_j| + |R_j| \\ &\leq (8k+1)|R_j| \\ &\leq (8k+1)ex(n, C_{2k}). \end{aligned}$$

Therefore

$$e(G) \leq \frac{(8k+1)(4k-1)}{2k}ex(n, C_{2k}) < (16k-2)ex(n, C_{2k}).$$

□

Proof of Theorem 2: By Lemmas A and B, we have that the number of triangles in G is at most $\frac{(2k-1)e(G)}{3} \leq \frac{(2k-1)(16k-2)}{3}ex(n, C_{2k})$.

□

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